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EQUIVALENCE RELATIONS INDUCED BY ACTIONS OF POLISH GROUPS

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ABSTRACT. We give an algebraic characterization of those sequences (H_n) of countable abelian groups for which the equivalence relations induced by Borel (or, equivalently, continuous) actions of $H_0 \times H_1 \times H_2 \times \cdots$ are Borel. In particular, the equivalence relations induced by Borel actions of H^ω, H countable abelian, are Borel iff $H \simeq \bigoplus_p (F_p \times \mathbb{Z}(p^\infty)^{n_p})$, where F_p is a finite p-group, $\mathbb{Z}(p^\infty)$ is the quasicyclic p-group, $n_p \in \omega$, and p varies over the set of all primes. This answers a question of R. L. Sami by showing that there are Borel actions of Polish abelian groups inducing non-Borel equivalence relations. The theorem also shows that there exist non-locally compact abelian Polish groups all of whose Borel actions induce only Borel equivalence relations. In the process of proving the theorem we generalize a result of Makkai on the existence of group trees of arbitrary height.

1. INTRODUCTION

Let G be a group acting on a set X. Put for $x, y \in X$

$$xE_G^X y \Leftrightarrow \exists g \in G \; gx = y.$$

Then $E_G^X \subset X \times X$ is an equivalence relation and is called the *equivalence relation induced by the action of* G *on* X. If G is Polish, X is a standard Borel space, and the action of G is Borel, then E_G^X is Σ_1^1. If additionally G is locally compact, then E_G^X is Borel. By Silver's theorem, it follows that the topological Vaught conjecture holds in this case; i.e., the action of G has either countably or "perfectly" many orbits. It was proved by R. L. Sami [S, Theorem 2.1] that the topological Vaught conjecture holds for Borel actions of abelian Polish groups. The proof, however, was different from the one in the locally compact case; in particular, it did not show that E_G^X was Borel for G Polish abelian.

The natural question was raised by Sami (see [S, p. 339]) whether E_G^X is Borel for all Borel (or, equivalently, continuous if X is a Polish space, see [BK]) actions of Polish abelian groups. We answer this question in the negative. We consider groups of the form $H_0 \times H_1 \times H_2 \times \cdots$ where the H_n's are countable. Such groups are equipped with the product topology (each H_n carrying the discrete topology) which is Polish and compatible with the group structure. We fully characterize those sequences (H_n) of countable abelian groups for which

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all Borel actions of $H_0 \times H_1 \times H_2 \times \cdots$ induce Borel equivalence relations. This happens precisely when all but finitely many of the H_n's are torsion and, for each prime p, for all but finitely many n's the p-component of H_n is of the form $F \times \mathbb{Z}(p^\infty)^m$, where F is a finite p-group, $\mathbb{Z}(p^\infty)$ is the quasicyclic p-group (i.e., $\mathbb{Z}(p^\infty) \simeq \{z \in \mathbb{C}\colon \exists n\ z^{p^n} = 1\}$), and $m \in \omega$. In particular, if $H_n = H$, $n \in \omega$, and H is countable abelian, then all Borel actions of $H \times H \times H \times \cdots$ induce Borel equivalence relations iff $H \simeq \bigoplus_p (F_p \times \mathbb{Z}(p^\infty)^{n_p})$, where F_p is a finite abelian p-group, $n_p \in \omega$, and p varies over the set of all primes. Thus, e.g., the group $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \cdots$ is abelian, Polish, and has a Borel action which induces a non-Borel equivalence relation. This answers Sami's question. On the other hand, $\mathbb{Z}(2^\infty) \times \mathbb{Z}(2^\infty) \times \mathbb{Z}(2^\infty) \times \cdots$ provides an interesting example of a Polish abelian group which is not locally compact but whose Borel actions induce only Borel equivalence relations. This shows that the implication "G locally compact $\Rightarrow E_G^X$ Borel" cannot be reversed. Some results for non-abelian H_n's are also obtained.

Now, we state some definitions and establish notation. By ω we denote the set of all natural numbers $\{0, 1, 2, \dots\}$. Ordinal numbers are identified with the set of their predecessors; in particular $n = \{0, 1, \dots, n-1\}$, for $n \in \omega$. By $\mathbb{Z}$, $\mathbb{Z}(p)$, $\mathbb{Z}(p^\infty)$, p a prime, we denote the group of integers, the cyclic group with p elements, and the quasicyclic p-group, respectively. By e we denote the identity element of a group and by $\langle X \rangle$, for a subset X of a group, the subgroup generated by X. We write $\langle h \rangle$ for $\langle \{h\} \rangle$. If H is a group, $\bigoplus_\omega H$ stands for the direct sum of countably many copies of H. A group H is called *p-compact* if for any decreasing sequence of groups $G_k < \mathbb{Z}(p) \times H$ with $\pi[G_k] = \mathbb{Z}(p)$, for each $k \in \omega$, we have $\pi[\bigcap_{k \in \omega} G_k] = \mathbb{Z}(p)$ where $\pi\colon \mathbb{Z}(p) \times H \to \mathbb{Z}(p)$ is the projection. If H is an abelian group and p is a prime, by the *p-component of* H we mean the maximal p-subgroup of H.

For a sequence of sets (H_n), $n \in \omega$, we write

$$H^n = H_0 \times \cdots \times H_{n-1}, \quad H^{<\omega} = \bigcup_{n \in \omega} H^n, \quad \text{and} \quad H^\omega = H_0 \times H_1 \times \cdots.$$

We also write A^ω for the product of infinitely many copies of A. If $x \in H^\omega$, put $lhx = \omega$; if $\sigma \in H^n$, some $n \in \omega$, put $lh\sigma = n$. For $\sigma \in H^{<\omega}$ and $x \in H^{<\omega} \cup H^\omega$, we write $\sigma * x$ for the concatenation of σ and x. If $x \in H^{<\omega} \cup H^\omega$ and $X \subset \omega$, we write $x|X$ for the unique element $y \in H^{<\omega} \cup H^\omega$ such that the domain of y is ω, if $X \cap lhx$ is infinite, and n, if $X \cap lhx$ is finite and has n elements, and $y(i) = x$ (the $(i+1)$th element of X). A set $S \subset H^{<\omega}$ is called a tree on (H_n) if $\sigma \in S$ implies $\sigma|n \in S$ for any $n < lh(\sigma)$. If S is a tree on (H_n) and $\sigma \in H^{<\omega}$, put $S_\sigma = \{\tau \in H^{<\omega}\colon \sigma * \tau \in S\}$. For a tree S on (H_n), H_n countable, define $S' = \{\sigma \in S\colon \exists \tau \in S\ \sigma \subset \tau, \sigma \neq \tau\}$. By transfinite induction define, for $\beta \in \omega_1$, $S^0 = S$ and $S^\beta = (S^\gamma)'$ if $\beta = \gamma + 1$, and $S^\beta = \bigcap_{\gamma < \beta} S^\gamma$ if β is limit. Put $\mathrm{ht}(S) = \min\{\beta\colon S^\beta = S^{\beta+1}\}$. For $\sigma \in H^{<\omega}$, put $r_S(\sigma) = \min\{\beta \in \omega_1\colon \sigma \notin S^\beta\}$ if there exists $\beta < \omega_1$ with $\sigma \notin S^\beta$, and $r_S(\sigma) = \omega_1$ otherwise. If there is no danger of confusion, we will omit the subscript in r_S. A tree on (H_n) is *well-founded* if there is no sequence $\sigma_i \in S$, $i \in \omega$, such that $\sigma_i \subset \sigma_{i+1}$ and $\mathrm{lh}(\sigma_i) \to \infty$ as $i \to \infty$. Now, assume that the H_n's are groups. The identity element $(e, e, \dots)$ of H^ω is denoted by $\vec{e}$. A tree S on (H_n) is called a *coset tree* if $S \cap H^n$ is a left coset of a subgroup of H^n for any $n \in \omega$; i.e., if $\sigma_1, \sigma_2, \sigma_3 \in S \cap H^n$, then $\sigma_1 \sigma_2^{-1} \sigma_3 \in S$. A coset tree S is called a *group tree* if $S \cap H^n$ is a subgroup of H^n for any $n \in \omega$. The notion of a group tree was introduced by Makkai in [M] and rediscovered by the author. We say that (H_n) *admits group (coset) trees of arbitrary height* if, for any $\beta < \omega_1$, there is a group (coset) tree T on (H_n) with $\mathrm{ht}(T) > \beta$. Let S be a coset tree on a sequence of groups (H_n). Then for each $n \in \omega$ there is a unique subgroup G_n of H^n which $S \cap H^n$ is a coset of. We actually have $G_n = \sigma^{-1}(S \cap H^n)$ for any $\sigma \in S \cap H^n$. Define

$$\alpha(S) = \bigcup_{n \in \omega} G_n.$$

Thus $\alpha(S) = \bigcup_{n \in \omega} \sigma_n^{-1}(S \cap H^n)$ where $\sigma_n \in S \cap H^n$ if $S \cap H^n \neq \varnothing$ and $\sigma_n = e$ otherwise. It is easy to see that $\alpha(S)$ is a group tree.

2. Main results

Theorem 1. *Let (H_n) be a sequence of countable abelian groups. Then the equivalence relation induced by any Borel action of H^ω is Borel iff, for all but finitely many n, H_n is torsion, and for all primes p for all but finitely many n the p-component of H_n is of the form $F \times \mathbb{Z}(p^\infty)^k$, where $k \in \omega$ and F is a finite abelian p-group.*

If H is countable, abelian, and torsion, then $H = \bigoplus_p H_p$, where p ranges over the set of all primes, and H_p is the p-component of H (see [F]). Thus we get the following corollary.

Corollary. *Let H be an abelian countable group. Then the equivalence relations induced by Borel actions of H^ω are Borel if and only if H is isomorphic to $\bigoplus_p (F_p \times \mathbb{Z}(p^\infty)^{n_p})$, where p ranges over the set of all primes, $n_p \in \omega$, and F_p is a finite abelian p-group.*

For not necessarily abelian countable groups, we have the following version of one implication from Theorem 1. (The definition of p-compactness is formulated in the introduction.)

Theorem 2. *Let (H_n) be a sequence of countable groups. If for each prime p, for all but finitely many n, H_n is p-compact, then the equivalence relations induced by Borel actions of H^ω are Borel.*

It is an open question whether the converse of Theorem 2 holds. This would be a natural extension of Theorem 1, since, as we show in Lemma 9, a countable abelian group is p-compact iff it is torsion and its p-component has the form as in Theorem 1.

Some of the ingredients of the proofs are: the theorem of Becker and Kechris [BK] on the existence of universal actions, the structure theory for countable abelian groups, and a construction of group trees of arbitrary height. It turns out that both conditions in Theorem 1 are equivalent to (H_n) not admitting group trees of arbitrary height (Lemma 12). This generalizes the known results that the sequence (H_n), $H_n = \mathbb{Z}$ for each $n \in \omega$, admits group trees of arbitrary height (Makkai [M, Lemma 2.6]), and that the sequence (H_n), $H_n = \bigoplus_\omega \mathbb{Z}(2)$ for each n, admits group trees of arbitrary height (Shelah [M, Appendix]). (See also [L, p. 979] for a proof of the latter result and its generalizations to groups which are direct sums of κ many copies of $\mathbb{Z}(2)$ for certain cardinals κ.) The known proofs in the above two cases—$\mathbb{Z}$ and $\bigoplus_\omega \mathbb{Z}(2)$—were different from each other, and Makkai's construction for $\mathbb{Z}$ rested on Dirichlet's theorem on primes in arithmetic progressions. We present a construction (Lemma 10) that encompasses both these cases and is purely combinatorial.

Here is how Theorems 1 and 2 follow from the lemmas in Sections 3–5. In Section 3, we prove that all Borel actions of H^ω, (H_n) a sequence of countable groups, induce Borel equivalence relations iff (H_n) does not admit well-founded coset trees of arbitrary height (Lemma 2). In Section 4, we show that (H_n) does not admit well-founded coset trees of arbitrary height iff it does not admit group trees of arbitrary height (Lemma 6). Then, in Section 5, we show that if for each prime p, for all but finitely many n, H_n is p-compact, then (H_n) does not admit group trees of arbitrary height (Lemma 8). This proves Theorem 2. Next, we prove that if (H_n) is a sequence of abelian groups, then (H_n) does not admit group trees of arbitrary height iff, for all but finitely many n, H_n is torsion and, for all primes p, for all but finitely many n, the p-component of H_n has the form as in Theorem 1 (Lemma 12). This proves Theorem 1.

3. Group actions and coset trees

The following construction is from [BK]. Let G be a Polish group. Consider $\mathscr{F}(G)$ the space of all closed subsets of G with the Effros Borel structure, i.e., the Borel structure generated by sets of the form $\{F \in \mathscr{F}(G) \colon F \cap V \neq \varnothing\}$ for $V \subset G$ open. Put $\mathscr{U}_G = \mathscr{F}(G)^\omega$, and define the following G-action on $\mathscr{U}_G \colon (g, (F_n)) \to (gF_n)$.

Theorem (Becker-Kechris [BK]). *$\mathscr{U}_G$ with the above G-action is a universal Borel G space; i.e., if X is a standard Borel space on which G acts by Borel automorphisms, then there is a Borel injection $\pi \colon X \to \mathscr{U}_G$ such that $\pi(gx) = g\pi(x)$ for $g \in G$ and $x \in X$.*

Let X be a standard Borel G-space. Let $\pi \colon X \to \mathscr{U}_G$ be a Borel injection whose existence is guaranteed by the above theorem. Then, for $x, y \in X$, we have

$$xE_G^X y \Leftrightarrow \pi(x) E_G^{\mathscr{U}_G} \pi(y).$$

This shows that the following corollary to the theorem above is true.

Lemma 1. *Let G be a Polish group. The relation induced by any Borel G-action is Borel iff the relation induced by the G-action on $\mathscr{U}_G$ is Borel.*

Lemma 2. *Let (H_n) be a sequence of countable groups. The equivalence relation induced by any Borel H^ω-action is Borel iff (H_n) does not admit well-founded coset trees of arbitrary height.*

Proof. Let $\mathscr{T}$ be the family of all trees on (H_n). The set $\mathscr{T}$ is a Polish space with the topology generated by sets of the form $\{T \in \mathscr{T} \colon \sigma \in T\}$ and $\{T \in \mathscr{T} \colon \sigma \notin T\}$ for $\sigma \in H^{<\omega}$.

($\Leftarrow$) By Lemma 1, it is enough to prove that the H^ω-action on $\mathscr{U}_{H^\omega}$ induces a Borel relation. Let $\mathscr{T}_p$ be the family of all pruned trees on (H_n), i.e., trees with no finite branches, with the topology inherited from $\mathscr{T}$. This topology makes $\mathscr{T}_p$ a Polish space. The mapping $\phi \colon \mathscr{T}_p \to \mathscr{F}(H^\omega)$ given by $\phi(T) = \{x \in H^\omega \colon \forall n \in \omega \;\; x|n \in T\}$ is a Borel isomorphism. For $x \in H^\omega$ and $T \in \mathscr{T}_p$ define

$$xT = \{\sigma \in H^{<\omega} \colon \sigma \in x|m(T \cap H^m) \text{ where } m = lh(\sigma)\}.$$

Then easily $xT \in \mathcal{T}_p$. Also $\phi(xT) = x\phi(T)$. Thus it is enough to check that the following action of H^ω on $\mathcal{T}_p{}^\omega$ induces a Borel equivalence relation: $(x, (T_n)) \to (xT_n)$, for $x \in H^\omega$, $(T_n) \in \mathcal{T}_p{}^\omega$.

Now define $\Phi: \mathcal{T}_p \times \mathcal{T}_p \to \mathcal{T}$ by

$$\Phi(T, S) = \{\sigma \in H^{<\omega}: T \cap H^m = \sigma(S \cap H^m) \text{ where } m = lh(\sigma)\}.$$

Easily $\Phi(T, S)$ is a coset tree. Define the mapping $\Psi: \mathcal{T}_p{}^\omega \times \mathcal{T}_p{}^\omega \to \mathcal{T}$ by

$$\Psi((T_n), (S_n)) = \bigcap_{n\in\omega} \Phi(T_n, S_n).$$

Note that the intersection of a family of coset trees is a coset tree. Thus, for any $(T_n), (S_n) \in \mathcal{T}_p{}^\omega$, $\Psi((T_n), (S_n))$ is a coset tree. Also note that

$$(T_n) E_{H^\omega}^{\mathcal{T}_p^\omega} (S_n) \Leftrightarrow \Psi((T_n), (S_n)) \text{ is not well-founded.}$$

Indeed, if $\sigma_0 \subset \sigma_1 \subset \cdots$, $lh(\sigma_i) \to \infty$, and $\sigma_i \in \Psi((T_n), (S_n))$, then $xS_n = T_n$ for each $n \in \omega$ where $x = \bigcup_{i\in\omega} \sigma_i$. If $xS_n = T_n$ for all $n \in \omega$ and some $x \in H^\omega$, then $x|i \in \Psi((T_n), (S_n))$ and $\{x|i: i \in \omega\}$ witnesses that $\Psi((T_n), (S_n))$ is not well-founded. Clearly Ψ is a Borel mapping. Thus, if we assume that there is $\beta \in \omega_1$ such that any well-founded coset tree on (H_n) has height $< \beta$, we get

$$(\mathcal{T}_p \times \mathcal{T}_p)\backslash E_{H^\omega}^{\mathcal{T}_p^\omega} = \Psi^{-1}(\{T \in \mathcal{T}: T \text{ well-founded and } \mathrm{ht}(T) < \beta\}).$$

But $\{T \in \mathcal{T}: T$ well-founded and $\mathrm{ht}(T) < \beta\}$ is Borel, whence $E_{H^\omega}^{\mathcal{T}_p^\omega}$ is Borel.

($\Rightarrow$) Assume (H_n) admits well-founded coset trees of arbitrary height. Define the following continuous action of H^ω on $\mathcal{T}$:

$$(x, T) \to xT = \{\sigma \in H^{<\omega}: \sigma \in x|m(T \cap H^m) \text{ where } m = lh(\sigma)\}.$$

Define a Borel function $\Phi_1: \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ by

$$\Phi_1(T, S) = \{\sigma \in H^{<\omega}: \forall m \leq lh(\sigma)\ T \cap H^m = \sigma|m(S \cap H^m)\}.$$

Now, if $E_{H^\omega}^{\mathcal{T}}$ is Borel, $\Phi_1[(\mathcal{T} \times \mathcal{T})\backslash E_{H^\omega}^{\mathcal{T}}]$ is $\mathbf{\Sigma}_1^1$. Also $\Phi_1[(\mathcal{T} \times \mathcal{T})\backslash E_{H^\omega}^{\mathcal{T}}] \subset \{T \in \mathcal{T}: T \text{ is well-founded}\}$. Since $\{T \in \mathcal{T}: T$ is well-founded$\}$ is a $\mathbf{\Pi}_1^1$ set and $T \to \mathrm{ht}(T)$ is a $\mathbf{\Pi}_1^1$-norm on it, by the boundedness principle, there is $\beta \in \omega_1$ such that, for any $T, S \in \mathcal{T}$, if $(T, S) \notin E_{H^\omega}^{\mathcal{T}}$, then $\mathrm{ht}(\Phi_1(T, S)) < \beta$. But note that if T is a coset tree, then $\Phi_1(T, \alpha(T)) = T$. Thus, for any well-founded coset tree T on (H_n), $\mathrm{ht}(T) = \mathrm{ht}(\Phi(T, \alpha(T))) < \beta$, a contradiction.

4. Coset and group trees

The next several lemmas lead to a proof that the existence of well-founded coset trees of arbitrary height is equivalent to the existence of group trees of arbitrary height (Lemma 6). We will use a few times the easy fact that $\{r(\sigma): \sigma \in T\} \supset \mathrm{ht}(T)$ for any tree T on (H_n).

Lemma 3. *Let S be a coset tree. Then:*

(i) $\alpha(S') = \alpha(S)'$;
(ii) *if $S^\xi \cap H^k \neq \varnothing$ for each $k \in \omega$, then* $\alpha(S^\xi) = \alpha(S)^\xi$.

Proof. To show (i), let $\sigma \in H^n$. Then $\sigma \in \alpha(S')$ implies that there are $\tau_1, \tau_2 \in S'$ such that $\sigma = \tau_1^{-1}\tau_2$. Now we can find $g, h \in H_n$ with $\tau_1 * g, \tau_2 * h \in S$. But then $\sigma * (g^{-1}h) = (\tau_1 * g)^{-1}(\tau_2 * h) \in \alpha(S)$. Thus $\sigma \in \alpha(S)'$. On the other hand, if $\sigma \in \alpha(S)'$, then there are $g \in H_n$ and $\tau_1, \tau_2 \in S$ with $\tau_1^{-1}\tau_2 = \sigma * g$. But then $\sigma = (\tau_1|n)^{-1}(\tau_2|n)$ and $\tau_1|n, \tau_2|n \in S'$, whence $\sigma \in \alpha(S')$.

Notice that if $S_n \supset S_{n+1}$, $n \in \omega$, are coset trees, and, for some $k \in \omega$, $\bigcap_{n\in\omega}(S_n \cap H^k) \neq \varnothing$, then $\alpha(\bigcap_{n\in\omega} S_n) \cap H^k = \bigcap_{n\in\omega} \alpha(S_n) \cap H^k$. To see this, pick $\sigma \in \bigcap_{n\in\omega} S_n \cap H^k$. Then

$$\alpha\left(\bigcap_{n\in\omega} S_n\right) \cap H^k = \sigma^{-1}\left(\bigcap_{n\in\omega} S_n \cap H^k\right) = \bigcap_{n\in\omega} \sigma^{-1}(S_n \cap H^k)$$
$$= \bigcap_{n\in\omega} \alpha(S_n) \cap H^k.$$

Using (i) and the above observation, we get (ii) by transfinite induction.

Lemma 4. *Let T be a group tree. Let $\sigma_n \in H^n$, $n \in \omega$, be such that $(\sigma_{n+1}|n)^{-1}\sigma_n \in T^\beta$ for some $\beta \in \omega_1$. Put $S = \bigcup_{n\in\omega} \sigma_n(T \cap H^n)$. Then S is a coset tree, and for any $\xi \leq \beta$ we have $S^\xi = \bigcup_{n\in\omega} \sigma_n(T^\xi \cap H^n)$.*

Proof. For $\xi \leq \beta$, define

$$S^{(\xi)} = \bigcup_{n\in\omega} \sigma_n(T^\xi \cap H^n).$$

In particular, $S^{(0)} = S$. First note that each $S^{(\xi)}$ is a coset tree. Indeed, if $m < n$, then $(\sigma_n|m)^{-1}\sigma_m \in T^\xi$. This follows easily by induction from our assumptions that it holds for $n = m + 1$ and the fact that T^ξ is a group tree. To check that $S^{(\xi)}$ is a tree, let $\tau \in T^\xi \cap H^n$. Then, for $m < n$, $(\sigma_n\tau)|m = (\sigma_n|m)(\tau|m) = \sigma_m(\sigma_m^{-1}(\sigma|m)(\tau|m)) \in S^{(\xi)} \cap H^m$ since $(\sigma_m^{-1}(\sigma_n|m))(\tau|m) \in T^\xi \cap H^m$. Thus $S^{(\xi)}$ is a tree, and because of the way it was defined, it is a coset tree. It is obvious that $\alpha(S^{(\xi)}) = T^\xi$ and that $\sigma_n \in S^{(\xi)}$ for any $n \in \omega, \xi \leq \beta$.

Now, we show by induction that, for $\xi \leq \beta$, $\alpha(S^\xi) = T^\xi$ and $\sigma_n \in S^\xi$ for each $n \in \omega$. Both statements are true for $\xi = 0$. If ξ is the limit and $\sigma_n \in S^\zeta$ for all $\zeta < \xi$, then clearly $\sigma_n \in S^\xi$. By Lemma 3(ii), we also have $\alpha(S^\xi) = \alpha(S)^\xi = T^\xi$. If ξ is a successor, say $\xi = \zeta + 1$, then, by Lemma 3(i) and the induction hypothesis, we get $\alpha(S^\xi) = \alpha(S^\zeta)' = (T^\zeta)' = T^\xi$. Since $\sigma_{n+1} \in S^\zeta$, $\sigma_{n+1}|n \in S^\xi$. Since $(\sigma_{n+1}|n)^{-1}\sigma_n \in T^\beta \subset T^\xi$, we have $\sigma_n = (\sigma_{n+1}|n)((\sigma_{n+1}|n)^{-1}\sigma_n) \in S^\xi$.

Thus $\alpha(S^{(\xi)}) = T^\xi = \alpha(S^\xi)$, i.e., for each $n \in \omega$, $S^{(\xi)} \cap H^n$ and $S^\xi \cap H^n$ are left cosets of the same subgroup of H^n. Also $(S^{(\xi)} \cap H^n) \cap (S^\xi \cap H^n) \neq \varnothing$, as σ_n belongs to the intersection. Thus we get $S^{(\xi)} \cap H^n = S^\xi \cap H^n$ for each $n \in \omega$, i.e., $S^{(\xi)} = S^\xi$.

Lemma 5. *Let T be a group tree with* $\mathrm{ht}(T) > \omega$. *Then there exist $\sigma_n \in H^n$ such that:*

(i) $(\sigma_{n+1}|n)^{-1}\sigma_n \in T$;
(ii) $\bigcup_{n\in\omega} \sigma_n(T \cap H^n)$ *is a well-founded tree of height* $< \omega \cdot 2$.

Proof. We start with the following observation. Let K be a countable group and let K_n, $n \in \omega$, be a strictly decreasing sequence of subgroups of K. Then there exist $g_n \in K$, $n \in \omega$, such that $g_n^{-1}g_{n+1} \in K_n$ and $\bigcap_{n\in\omega} g_nK_n = \varnothing$. To see that this is true, enumerate $K = \{k_n : n \in \omega\}$ and pick $g_n \in K$ recursively so that $g_{n+1}K_{n+1} \subset g_nK_n$ and $k_n \notin g_{n+1}K_{n+1}$.

Now, assume that T is a group tree and $\mathrm{ht}(T) > \omega$. Let σ_0 be such that $r(\sigma_0) = \omega$. Put $k_0 = lh(\sigma_0) + 1$. Then $\{r(\sigma) : \sigma \in T \cap H^{k_0}\} \cap \omega$ is cofinal in ω. Let $p_n : H^n \to H^{k_0}$, $n > k_0$, denote the projection on the first k_0 coordinates. Since $\{\sigma \in H^{k_0} : r(\sigma) \geq m\} = p_{k_0+m}[T \cap H^{k_0+m}]$, there is an increasing sequence $k_0 < m_0 < m_1 < m_2 < \cdots$ such that $p_{m_{n+1}}[T \cap H^{m_{n+1}}] \neq p_{m_n}[T \cap H^{m_n}]$ and, obviously, $p_{m_{n+1}}[T \cap H^{m_{n+1}}] \subset p_{m_n}[T \cap H^{m_n}]$. Pick $\tau_n \in H^{k_0}$, $n \in \omega$, as in the preceding paragraph for $K_n = p_{m_n}[T \cap H^{m_n}]$, i.e.,

$$\tau_{n+1}^{-1}\tau_n \in p_{m_n}[T \cap H^{m_n}] \quad \text{and} \quad \bigcap_{n\in\omega} \tau_n(p_{m_n}[T \cap H^{m_n}]) = \varnothing.$$

We recursively construct $\sigma_n \in H^n$, $n \in \omega$, so that $\sigma_{m_n}|k_0 = \tau_n$ and $(\sigma_{n+1}|n)^{-1}\sigma_n \in T$. First, find $\rho_n \in H^{m_n}$ so that $\rho_n|k_0 = \tau_n$ and $(\rho_{n+1}|m_n)^{-1}\rho_n \in T$. For ρ_0 take any extension of τ_0 in H^{m_0}. Now assume ρ_n has been constructed. Then $\tau_{n+1}^{-1}(\rho_n|k_0) = \tau_{n+1}^{-1}\tau_n \in p_{m_n}[T \cap H^{m_n}]$. Let $\sigma \in T \cap H^{m_n}$ be such that $\tau_{n+1}^{-1}(\rho_n|k_0) = \sigma|k_0$. Note that $(\rho_n\sigma^{-1})|k_0 = \tau_{n+1}$, and let ρ_{n+1} be an arbitrary extension of $\rho_n\sigma^{-1}$ in $H^{m_{n+1}}$. Now, put $\sigma_n = \rho_l|n$ if $0 \leq n \leq m_0$ and $l = 0$ or if $m_{l-1} < n \leq m_l$ and $l > 0$.

We have $\sigma_{m_n}|k_0 = \rho_n|k_0 = \tau_n$. Also $(\sigma_{n+1}|n)^{-1}\sigma_n \in T$, i.e., (i) is easy to see. Put $S = \bigcup_{n\in\omega} \sigma_n(T \cap H^n)$. To check (ii), let $\sigma \in S \cap H^{k_0}$. Pick the unique $k \in \omega$ such that $\sigma \in \tau_k(p_{m_k}[T \cap H^{m_k}]) \setminus \tau_{k+1}(p_{m_{k+1}}[T \cap H^{m_{k+1}}])$. Then for any $\sigma' \in S$ with $\sigma' \supset \sigma$, we have $lh\sigma' < m_{k+1}$. Otherwise, $\sigma' \in \sigma_n(T \cap H^n)$ for some $n \geq m_{k+1}$, whence $\sigma = p_n(\sigma') \in \tau_n(p_n[T \cap H^n])$, a contradiction. Thus $r_S(\sigma) < \omega$ for any $\sigma \in S \cap H^{k_0}$. It follows that S is well-founded and $\mathrm{ht}(S) \leq \omega + k_0$.

Lemma 6. *Let (H_n) be a sequence of countable groups. Then the following conditions are equivalent:*

(i) *(H_n) admits well-founded coset trees of arbitrary height;*
(ii) *(H_n) admits coset trees of arbitrary height;*
(iii) *(H_n) admits group trees of arbitrary height.*

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). Note that if $S \subset T$ are coset trees and $S \neq T$, then $\alpha(S) \subset \alpha(T)$ and $\alpha(S) \neq \alpha(T)$. To see this, pick $k \in \omega$ such that $S \cap H^k \neq T \cap H^k$ and $\sigma \in S \cap H^k$. Then $\alpha(S) \cap H^k = \sigma^{-1}(S \cap H^k) \neq \sigma^{-1}(T \cap H^k) = \alpha(T) \cap H^k$.

Now, let S be a given coset tree. Define

$$\gamma = \min\{\min\{\xi : \exists k\ S^\xi \cap H^k = \varnothing\}, \mathrm{ht}(S)\}.$$

Then, by Lemma 3(ii) and the above observation, we have $\alpha(S)^\xi = \alpha(S^\xi) \neq \alpha(S^\zeta) = \alpha(S)^\zeta$ for $\xi < \zeta < \gamma$, whence $\mathrm{ht}(\alpha(S)) \geq \gamma$. But it is easy to see that $\mathrm{ht}(S) < \gamma + \omega$. Thus (ii) $\Rightarrow$ (iii) is proved.

(iii) $\Rightarrow$ (i). Let T be a group tree of height $> \beta + \omega$. We show that there is a well-founded coset tree of height $\geq \beta$. To this end consider T^β. Then $\mathrm{ht}(T^\beta) > \omega$. Apply Lemma 5 to T^β to find $\sigma_n \in H^n$, $n \in \omega$, as in Lemma 5(i) and (ii). Put $S = \bigcup_{n\in\omega} \sigma_n(T \cap H^n)$. Then, by Lemma 4, S is a coset tree and $S^\beta = \bigcup_{n\in\omega}\sigma_n(T^\beta\cap H^n) \neq \varnothing$. By Lemma 5(ii), $S^{\beta+\omega\cdot 2} = (\bigcup_{n\in\omega}\sigma_n(T^\beta\cap H^n))^{\omega\cdot 2} = \varnothing$. Thus S is a well-founded tree with $\mathrm{ht}(S) \geq \beta$.

5. Group trees and algebraic properties of groups

Lemma 7. *Let H be a countable group. If H is not torsion, it is not p-compact for any prime p.*

Proof. Clearly, if a subgroup of H is not p-compact, neither is H. Thus it is enough to show that $\mathbb{Z}$ is not p-compact. This is witnessed by the following sequence of subgroups of $\mathbb{Z}(p)\times\mathbb{Z}$:

$$G_k = \{(m(p+1)^k \bmod p,\, m(p+1)^k) : m\in\mathbb{Z}\}, \qquad k\in\omega.$$

Lemma 8. *Let (H_n) be a sequence of countable groups. If (H_n) admits group trees of arbitrary height, then there exist a prime p and infinitely many $n\in\omega$ such that H_n is not p-compact.*

Proof. If, for infinitely many $n\in\omega$, H_n is not torsion, we are done by Lemma 7. Also, if (H_n) admits group trees of arbitrary height, so does $(H_n)_{n\geq N}$ for any $N\in\omega$. This follows from Lemma 6 as soon as we notice that if S is a coset tree on (H_n) and $\sigma\in H^N$, then S_σ is a coset tree on $(H_n)_{n\geq N}$, and that, given $\beta<\omega_1$, if $\mathrm{ht}(S)$ is large enough, then $\mathrm{ht}(S_\sigma)>\beta$ for some $\sigma\in H^N$. Thus, we can assume that H_n is torsion for each n, and that there exists a group tree on (H_n) of height $>\omega^2$.

Let T be a group tree on (H_n). Let p be a prime. Assume $\sigma\in T\cap H^n$, $r(\sigma)<\omega_1$, and the order of σ is a power of p. Let $\beta<r(\sigma)$. Then there is $\tau\supset\sigma$ such that $r(\tau)=\beta$ and the order of τ is a power of p. To see this, let $\tau'\supset\sigma$, $\tau'\neq\sigma$ and $r(\tau')\geq\beta$. Let $l\in\omega$ be such that p does not divide it and the order of $l\tau'$ is a power of p. Since the order of σ is a power of p, there is $l'\in\omega$ such that $l'l\sigma=\sigma$. Put $\tau_1=l'l\tau'$. Note that $\tau_1\supset\sigma$ and $\tau_1\neq\sigma$. Since, for any $\gamma\in\omega_1$ and $m\in\omega$, $\{\tau\in T\cap H^m : r(\tau)\geq\gamma\}$ is a subgroup of H^m (this follows easily from the facts that $\{\tau\in T\cap H^m : r(\tau)\geq\gamma\} = T^\gamma\cap H^m$ and that T^γ is a group tree), $r(\tau_1)=r(l'l\tau')\geq r(\tau')\geq\beta$. If $r(\tau_1)=\beta$, we are done. If $r(\tau_1)>\beta$, we repeat the above construction and get $\tau_2\supset\tau_1$, $\tau_2\neq\tau_1$, whose order is a power of p and $r(\tau_2)\geq\beta$. Again, if $r(\tau_2)=\beta$, we are done; otherwise we repeat the construction. Note that we cannot do it indefinitely, since then we would produce a sequence $\omega\subset\tau_1\subset\tau_2\subset\cdots$, $\tau_m\neq\tau_{m+1}$, whence $r(\sigma)=\omega_1$, a contradiction. Thus we must obtain $\tau_m\supset\sigma$ such that $r(\tau_m)=\beta$ and the order of τ_m is a power of p.

Next, notice that if $\tau\in T\cap H^n$, $r(\tau)$ is a limit, and the order of τ is a power of p, p a prime, then H_n is not p-compact. Indeed, let γ_k, $k\in\omega$, be a strictly increasing sequence of ordinals tending to $r(\tau)$. Put $G_k=\{\sigma\in T\cap H^{n+1} : r(\sigma)\geq\gamma_k\}$. Let $\pi : H^{n+1}\to H^n$ be the projection. Notice that (G_k) is a decreasing sequence of subgroups of H^{n+1} and $\tau\in\bigcap_{k\in\omega}\pi[G_k]\setminus\pi[\bigcap_{k\in\sigma}G_k]$. Let $C=\langle\tau\rangle$. Then $C<H^n$ and $C\simeq\mathbb{Z}(p^m)$ for some $m\in\omega$. Put $G'_k = G_k\cap(C\times H_n)$. Let $\phi : C\to\mathbb{Z}(p)$ be a surjective homomorphism. Let $\Phi = \phi\times\mathrm{id} : C\times H_n\to\mathbb{Z}(p)\times H_n$. Since Φ is finite-to-1, $\Phi[\bigcap_{k\in\omega}G'_k] = \bigcap_{k\in\omega}\Phi[G'_k]$. Note also that $\pi'\circ\Phi=\phi\circ\pi$ where $\pi' : \mathbb{Z}(p)\times H_n\to\mathbb{Z}(p)$ is the projection.

Thus

$$\phi\left[\pi\left[\bigcap_{k\in\omega} G'_k\right]\right] = \pi'\left[\bigcap_{k\in\omega} \Phi[G'_k]\right].$$

But $\pi[\bigcap_{k\in\omega} G'_k] \neq C$ whence $\pi[\bigcap_{k\in\omega} G'_k] \subset \ker(\phi)$. Thus $\phi[\pi[\bigcap_{k\in\omega} G'_k]] = \{0\}$ and finally

$$\pi'\left[\bigcap_{k\in\omega} \Phi[G'_k]\right] = \{0\}.$$

On the other hand,

$$\bigcap_{k\in\omega} \pi'[\Phi[G'_k]] = \phi\left[\bigcap_{k\in\omega} \pi[G'_k]\right] = \mathbb{Z}(p).$$

Thus the decreasing sequence of groups $\Phi[G'_k]$, $k \in \omega$, witnesses that H_n is not p-compact.

Now, let T be a group tree on (H_n) with $\mathrm{ht}(T) > \omega^2$. There exists a prime p and $\sigma \in T$ such that the order of σ is a power of p and $\omega^2 \leq r(\sigma) < \omega_1$. To show this, first find $\tau \in T$ with $r(\tau) = \omega^2$. The group $G = \langle \tau \rangle$ is cyclic and finite. Thus there are $\sigma_1, \sigma_2, \dots, \sigma_m \in T \cap H^n$, $n = lh(\tau)$, which commute with each other, their orders are powers of distinct primes and $\tau = \sigma_0 \cdots \sigma_m$. Note that for each $0 \leq i \leq m$ there is $k \in \omega$ with $k\tau = \sigma_i$. Thus, since $\{\sigma \in T \cap H^n : r(\sigma) \geq \omega^2\}$ is a subgroup of H^n, $r(\sigma_i) \geq \omega^2$ for all $0 \leq i \leq m$. Also $\{\sigma \in T \cap H^n : r(\sigma) \geq \omega_1\}$ is a subgroup of H^n, thus there is i such that $r(\sigma_i) < \omega_i$, and we are done.

Now, fix the prime p and $\sigma \in T$ as above. Let $N \in \omega$. We show that there are more than N numbers n such that H_n is not p-compact. Indeed, we can recursively produce $\tau_0, \tau_1, \dots, \tau_N \in T$ so that $\sigma \subset \tau_0$ and $r(\tau_0) = \omega^2$, $\tau_i \subset \tau_{i+1}$, the order of each τ_i is a power of p, and $r(\tau_i) = \omega \cdot (N + 1 - i)$, $1 \leq i \leq N$. But then if we put $n_i = lh(\tau_i)$, we get $n_0 < n_1 < \cdots < n_N$ and H_{n_i} is not p-compact since ω^2 and $\omega \cdot (N + 1 - i)$, $1 \leq i \leq N$, are limit.

In the following lemma, we essentially find all abelian countable groups which are p-compact.

Lemma 9. *Let H be an abelian countable group. Let p be a prime. Then the following conditions are equivalent*:

(i) *H is p-compact*;

(ii) *H is torsion, and the p-component of H is of the form $F \times \mathbb{Z}(p^\infty)^n$ where F is a finite p-group and $n \in \omega$*;

(iii) *H is torsion, and there is no surjective homomorphism mapping a subgroup of H onto $\bigoplus_\omega \mathbb{Z}(p)$*.

Proof. (ii) $\Rightarrow$ (i). Let $G_k < \mathbb{Z}(p) \times H$, $k \in \omega$, $G_{k+1} < G_k$, and $\pi[G_k] = \mathbb{Z}(p)$ where $\pi: \mathbb{Z}(p) \times H \to \mathbb{Z}(p)$ is the projectin. Now, $H = H_p \times H'$ and $G_k = (G_k)_p \times G'_k$ where H_p and $(G_k)_p$ are the p-components of H and G_k, respectively, and the order of any element of H' or G'_k is not divisible by p [F, Theorem 8.4]. Clearly we have $(G_k)_p < \mathbb{Z}(p) \times H_p$. We say that a group fulfils the minimum condition if each strictly decreasing sequence of subgroups is finite. Since, as one can easily see, $\mathbb{Z}(p^\infty)$ and finite groups fulfil the minimum condition, and the property of fulfilling the minimum condition is preserved under taking finite products, $\mathbb{Z}(p) \times H_p$ fulfils the minimum condition. Thus there is $k_0 \in \omega$ such that $(G_k)_p = (G_{k_0})_p$ for $k \geq k_0$. But then

$$\pi\left[\bigcap_{k\in\omega} G_k\right] = \pi\left[\bigcap_{k\in\omega}(G_k)_p \times \bigcap_{k\in\omega} G'_k\right] \supset \pi[(G_{k_0})_p \times \{0\}]$$
$$= \pi[(G_{k_0})_p \times G'_{k_0}] = \pi[G_{k_0}] = \mathbb{Z}(p).$$

(i) $\Rightarrow$ (iii). By Lemma 7, H is torsion. Note that if F_1 can be mapped by a homomorphism onto F_2, F_1, F_2 groups, and F_2 is not p-compact, then F_1 is not p-compact either. Indeed, let $\phi\colon F_1 \to F_2$ be a surjective homomorphism, and let the sequence (G_k) of subgroups of $\mathbb{Z}(p) \times F_2$ witness that F_2 is not p-compact, then

$$G'_k = \{(m, g) \in \mathbb{Z}(p) \times F_1 \colon (m, \phi(g)) \in G_k\}$$

witness that F_1 is not p-compact. Thus to prove that H is not p-compact, assuming (iii) fails, it is enough to show that $\bigoplus_\omega \mathbb{Z}(p)$ is not p-compact. Let $\{e_i\colon i \in \omega\}$ be an independent set generating $\bigoplus_\omega \mathbb{Z}(p)$. Let us fix a sequence of sets $X_k \subset \omega$, $k \in \omega$, such that $X_{k+1} \subset X_k$ and $\bigcap_{k\in\omega} X_k = \varnothing$. Define $G_k < \mathbb{Z}(p) \times \bigoplus_\omega \mathbb{Z}(p)$ by

$$G_k = \langle\{(m, me_i)\colon i \in X_k, m \in \mathbb{Z}(p)\}\rangle.$$

Then (G_k) witnesses that $\bigoplus_\omega \mathbb{Z}(p)$ is not p-compact.

(iii) $\Rightarrow$ (ii). Assume (iii). Let H_p be the p-component of H. Let $H_p^1 = \bigcap_{n\in\omega} nH_p$ be its first Ulm group. If H_p/H_p^1 is infinite, then

$$H_p/H_p^1 \simeq \bigoplus_{m\in\omega} \mathbb{Z}(p^{n_m})$$

for a sequence $n_m \in \omega\backslash\{0\}$ [F, Theorem 17.2 and remarks on p. 155]. Thus H_p/H_p^1, and hence H_p, can be mapped homomorphically onto $\bigoplus_\omega \mathbb{Z}(p)$. Therefore H_p/H_p^1 is finite. Put $F = H_p/H_p^1$. But then H_p^1 is divisible [F, Lemma 37.2] and $H_p \simeq F \times H_p^1$ [F, Theorem 21.2]. Now, by [F, Theorem 23.1], either $H_p^1 \simeq \mathbb{Z}(p^\infty)^n$, for some $n \in \omega$, and we are done, or $H_p^1 \simeq \bigoplus_\omega \mathbb{Z}(p^\infty)$. But in the latter case H_p^1, and hence H, contain an isomorphic copy of $\bigoplus_\omega \mathbb{Z}(p)$, a contradiction

Remark. (In this remark the notation and terminology follow [F].) One can give other characterizations of p-compactness among countable torsion abelian groups. For example p-compactness of H is equivalent to the following conditions:

(iv) the p-component of H fulfills the minimum condition;

(v) for any finite p-group $F < H$ the p-rank of H/F is finite.

Obviously (ii) $\Rightarrow$ (iv), and (iv) $\Rightarrow$ (i) as in the proof of (ii) $\Rightarrow$ (i). Now, assuming (iv) and noticing that a homomorphic image of a group fulfilling the minimum condition fulfills the minimum condition, we get that the p-component of H/F, $F < H$ finite, fulfills the minimum condition. This obviously implies that its p-rank is finite. Thus (iv) $\Rightarrow$ (v). To see (v) $\Rightarrow$ (ii), let H_p be the p-component of H. Let τ be its Ulm type. First note that if $\tau = \gamma+1$, for some γ, then H_p^γ/H_p^τ is finite. Otherwise, $r_p(H_p^\gamma/H_p^\tau) = \infty$, and since $H_p \simeq H_p/H_p^\tau \times H_p^\tau$, we get $r_p(H_p) = \infty$. Now, we claim that τ is neither a limit ordinal nor a successor of a limit ordinal. Otherwise, using the above observation there is a sequence of groups $G_n < H_p/H_p^\tau$, $n \in \omega$, such that $G_{n+1} < G_n$, $G_{n+1} \neq G_n$ and $\bigcap_{n\in\omega} G_n$ is finite. Put $G = \bigcap_{n\in\omega} G_n$. Then we can pick recursively $g_k \in H_p/H_p^\tau$ so that $pg_k \in G$ and for each k there is an n with $g_k \in G_n$ and $g_i \notin G_n$ for $i < k$. Then clearly the image of $\{g_k : k \in \omega\}$ under the natural homomorphism $H_p/H_p^\tau \to (H_p/H_p^\tau)/G$ is infinite independent. Again, since $H_p \simeq H_p/H_p^\tau \times H_p^\tau$, $r_p(H/G') = r_p(H_p/G') = \infty$ for some finite p-group G'. Next, notice that τ is not of the form $\gamma + 2$ because in this case $H_p^{\gamma+1}/H_p^\tau$ is finite and $r_p(H_p^\gamma/H_p^{\gamma+1}) = \infty$ whence $r_p((H_p^\gamma/H_p^\tau)/(H_p^{\gamma+1}/H_p^\tau)) = \infty$. And as before $r_p(H/G') = \infty$ for some finite p-group G'. Thus $\tau \leq 1$, and if $\tau = 1$, then H_p/H_p^1 is finite. If $\tau = 0$, H_p is divisible, and since $r_p(H_p) < \infty$, there is $n \in \omega$ with $H_p \simeq \mathbb{Z}(p^\infty)^n$. If $\tau = 1$, put $F = H_p/H_p^1$. Then $H_p \simeq F \times H_p^1$, F finite, H_p^1 divisible. Since $r_p(H_p^1) < \infty$, there is $n \in \omega$ with $H_p^1 \simeq \mathbb{Z}(p^\infty)^n$.

Now, we make a technical definition useful in proving the existence of group trees of arbitrary height. An abelian countable group H is called *manageable* if there exist two decreasing sequences of subgroups (G_n^0), (G_n^1) with $\bigcap_{n\in\omega} G_n^i = \{e\}$, for $i = 0, 1$, and a homomorphism $\phi: H \times H \to H$ such that $\phi[G_n^0 \times G_n^1] = H$ for any $n \in \omega$.

Lemma 10. *Let H be a countable abelian group. If H is manageable, then (H_n), where $H_n = H$ for each $n \in \omega$, admits group trees of arbitrary height.*

Proof. Fix two decreasing sequences of subgroups (G_n^0) and (G_n^1) and a homomorphism ϕ as in the definition of manageability. For each ordinal $\beta < \omega_1$, we produce a group tree T_β such that:

— if $\beta = \gamma + 1$, then $T_\beta^\gamma \cap H = H$ and $\forall h \in H$ ($h \neq e \Rightarrow (T_\beta)_h$ is well-founded);

— if β is limit, then $\forall \gamma < \beta\ \exists n \in \omega\ (T_\beta^\gamma \cap H^2 \supset G_n^0 \times G_n^1)$ and $\forall \sigma \in H^2$ ($\sigma \neq (e, e) \Rightarrow (T_\beta)_\sigma$ is well-founded).

Then clearly $\omega_1 > r_{T_\beta}(h) \geq \beta$ for any $h \in H\backslash\{e\}$ in the first case, and for any $\gamma < \beta$, $\omega_1 > r_{T_\beta}(\sigma) \geq \gamma$ for some $\sigma \in H^2\backslash\{(e, e)\}$ in the latter. Thus $\mathrm{ht}(T_\beta) \geq \beta$ for any $\beta \in \omega_1$.

Put $T_0 = \{\vec{e}\}$ and $T_1 = H \cup \{\vec{e}\}$. Assume T_γ has been defined for all $\gamma < \beta$. If $\beta = \gamma + 1$ and γ is a successor, put

$$T_\beta = \{\varnothing\} \cup H \cup \{\sigma(0) * \sigma : \sigma \in T_\gamma, lh\sigma \geq 1\}.$$

If $\beta = \gamma + 1$ and γ is a limit, put

$$T_\beta = \text{the tree generated by } \{\phi(\sigma(0), \sigma(1)) * \sigma : \sigma \in T_\gamma, lh\sigma \geq 2\}.$$

Checking that the T_β's work is straightforward. Now, assume β is a limit ordinal. Note that it is enough to construct two group trees S_0 and S_1 such that there is an increasing sequence $\gamma_n \to \beta$ with $S_0^{\gamma_n} \cap H \supset G_n^0$ and $S_1^{\gamma_n} \cap H \supset G_n^1$ and $\forall h \in H$ ($h \neq e \Rightarrow (S_0)_h$ and $(S_1)_h$ are well-founded). If S_0 and S_1 are defined, let

$$T_\beta = \{\sigma \in H^{<\omega} : \sigma|\{2k : k \in \omega\} \in S_0 \text{ and } \sigma|\{2k+1 : k \in \omega\} \in S_1\}.$$

We will define a group tree $S = S_0$ as above; the construction of S_1 is analogous. Put $G_n^0 = G_n$. Fix an increasing sequence of successors $\gamma_n \to$

β, $n \in \omega$. Find pairwise disjoint infinite sets X_n, $n \in \omega$, with $\bigcup_{n\in\omega} X_n = \omega$. Let

$$R_n = \{\varnothing\} \cup \{h * \sigma \colon h \in G_n,\ \sigma|X_n \in T_{\gamma_n},\ \sigma|(\omega\backslash X_n) \subset \vec{e}, \text{ and if } lh\sigma > \min X_n, \text{ then } h = (\sigma|X_n)(0)\}.$$

Note that each R_n is a group tree. Define

$$S = \bigcup_{k\in\omega} \langle H^k \cap \bigcup_{n\in\omega} R_n \rangle.$$

Easily S is a group tree. To see $S^{\gamma_n} \cap H \supset G_n$, just notice that, for each $h \in G_n$, $r_{T_{\gamma_n}}(h) \geq \gamma_n$, and there is a monotone 1-to-1 mapping $\psi\colon (T_{\gamma_n})_h \to S$ defined by $\psi(\sigma) = h * \tau$, where $\tau \in H^{<\omega}$ is maximal such that $\tau|X_n = h * \sigma$ and $\tau|(\omega\backslash X_n) \subset \vec{e}$. To show that $(S)_h$ is well-founded for $h \in H\backslash\{e\}$, fix $h \in H$ with $h \neq e$, and assume towards a contradiction that $h * x$ is an infinite branch through S for some $x \in H^\omega$. Find $n \in \omega$ with $h \notin G_n$. Let $k \in \omega$ be such that $k \cap X_i \neq \varnothing$ for $i \in n$. Put $\tau = x|k$ and $n_i = \min X_i$ for $i \in n$. If $\tau(n_{i_0}) \neq e$ for some $i_0 \in n$, notice that $x|X_{i_0}$ is an infinite branch through $T_{\gamma_{i_0}}$ with $(x|X_{i_0})(0) \neq e$ which contradicts the inductive assumption. Thus we can assume that $\tau(n_i) = e$ for all $i \in n$. Then, since the R_i's are group trees, $h * \tau = \sigma \cdot \prod_{i\in n}(h_i * \tau_i)$ for some $\sigma \in G_n \times H^k$ with $\sigma(n_i) = e$ and some $h_i * \tau_i \in R_i \cap H^{k+1}$. By the definition of R_i, $h_i = \tau_i(n_i) = \tau(n_i) = e$. Thus $h = \sigma(0) \in G_n$, a contradiction.

Lemma 11. *Let (H_n) be a sequence of countable groups. Then (H_n) admits group trees of arbitrary height if either of the following conditions holds.*

(i) *There exists a sequence $n_0 < n_1 < \cdots$ such that (H_{n_k}) admits group trees of arbitrary height.*

(ii) *For each n, G_n is a homomorphic image of a subgroup of H_n, and (G_n) admits group trees of arbitrary height.*

Proof. (i) Let T be a group tree on (H_{n_k}). Define $\overline{T}$ a group tree on (H_n) as follows:

$$\sigma \in \overline{T} \quad \text{iff} \quad \sigma|X \in T \text{ and } \sigma|(\omega\backslash X) = \vec{e}|(\omega\backslash X)$$

where $X = \{n_k \colon k \in \omega\}$. Then $\text{ht}(\overline{T}) \geq \text{ht}(T)$.

(ii) Fix $H'_n < H_n$ and surjective homomorphisms $\phi_n\colon H'_n \to G_n$. Let T be a group tree on (G_n). Define $\overline{T}$ a group tree on (H_n) as follows:

$$\sigma \in \overline{T} \quad \text{iff} \quad \forall k < lh\sigma\ (\sigma(k) \in H'_n \text{ and } (\phi_0(\sigma(0)), \ldots, \phi_k(\sigma(k))) \in T).$$

Then $\text{ht}(\overline{T}) \geq \text{ht}(T)$.

Lemma 12. *Let (H_n) be a sequence of countable abelian groups. Then (H_n) does not admit group trees of arbitrary height iff H_n is torsion for all but finitely many n, and for each prime p, for all but finitely many n the p-component of H_n is of the form $F \times \mathbb{Z}(p^\infty)^k$, where F is a finite p-group, $k \in \omega$.*

Proof. The implication $\Leftarrow$ follows from Lemmas 8 and 9. To see $\Rightarrow$, assume the conclusion does not hold. Then either there exist infinitely many n such that H_n contains an isomorphic copy of $\mathbb{Z}$ or, by Lemma 9, there exist a prime p and infinitely many n such that a subgroup of H_n can be mapped homomorphically onto $\bigoplus_\omega \mathbb{Z}(p)$. Thus, by Lemma 11, it is enough to show that (H_n), where $H_n = \mathbb{Z}$ for each n or $H_n = \bigoplus_\omega \mathbb{Z}(p)$ for each n, admits group trees of arbitrary height.

By Lemma 10, it suffices to prove that $\mathbb{Z}$ and $\bigoplus_\omega \mathbb{Z}(p)$ are manageable. For $\mathbb{Z}$, put $G_n^0 = \langle 2^n \rangle$, $G_n^1 = \langle 3^n \rangle$. Define $\phi\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $\phi(m, l) = m + l$. For $\bigoplus_\omega \mathbb{Z}(p)$, fix an infinite independent set $\{e_i\colon i \in \omega\}$ generating $\bigoplus_\omega \mathbb{Z}(p)$. Find a decreasing sequence of nonempty sets $X_n \subset \omega$, $n \in \omega$, such that $\bigcap_{n\in\omega} X_n = \varnothing$. Put $G_n^0 = \langle \{e_i\colon i \in X_n\} \rangle$ and $G_n^1 = \{e\}$. Fix a function $f\colon \omega \to \omega$ so that, for any $n, m \in \omega$, $f^{-1}(m) \cap X_n \neq \varnothing$. Define $\phi'\colon \bigoplus_\omega \mathbb{Z}(p) \to \bigoplus_\omega \mathbb{Z}(p)$ to be the unique homomorphism extending $\phi'(e_i) = e_{f(i)}$. Let $\phi\colon \bigoplus_\omega \mathbb{Z}(p) \times \bigoplus_\omega \mathbb{Z}(p) \to \bigoplus_\omega \mathbb{Z}(p)$ be the composition of the projection to the first coordinate with ϕ'.

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